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$S_3 \times \mathbb{Z}_2$ model for neutrino mass matrices

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Abstract: We propose a model for lepton mass matrices based on the seesaw mechanism, a complex scalar gauge singlet and a horizontal symmetry $S_3 \times \mathbb{Z}_2$. In a suitable weak basis, the charged-lepton mass matrix and the neutrino Dirac mass matrix are diagonal, but the vacuum expectation value of the scalar gauge singlet renders the Majorana mass matrix of the right-handed neutrinos non-diagonal, thereby generating lepton mixing. When the symmetry S_3 is not broken in the scalar potential, the effective light-neutrino Majorana mass matrix enjoys μ-τ interchange symmetry, thus predicting maximal atmospheric neutrino mixing together with $U_{e3} = 0$. A partial and less predictive form of μ-τ interchange symmetry is obtained when the symmetry S_3 is softly broken in the scalar potential. Enlarging the symmetry group $S_3 \times \mathbb{Z}_2$ by an additional discrete electron-number symmetry $\mathbb{Z}_2^{(e)}$, a more predicitive model is obtained, which is in practice indistinguishable from a previous one based on the group D_4.

Keywords: Global Symmetries, Solar and Atmospheric Neutrinos, Neutrino Physics.

Contents

1. Introduction

The very precise data now existing on neutrino mass-squared differences and on lepton mixing [1], and the prospects of rapid experimental developments in this field, invite theorists to construct models for the lepton mass matrices, in an effort to exploit and to understand the symmetries and hierarchies suggested by the data. Among them, most prominent are the possible maximality of the atmospheric neutrino mixing angle θ_{23} and the smallness of the mixing-matrix element U_{e3}. Together, they suggest the existence of a μ-τ interchange symmetry in the (effective) light-neutrino Majorana mass matrix $\mathcal{M}_\nu$, taken in the basis where the charged-lepton mass matrix is diagonal [2]. Such a symmetry, embodied in

$$\mathcal{M}_\nu = \begin{pmatrix} x & y & y \\ y & z & w \\ y & w & z \end{pmatrix}, \tag{1.1}$$

automatically leads to, simultaneously, $U_{e3} = 0$ and $\theta_{23} = \pi/4$. Various authors have dwelt on the matrix (1.1), and on generalizations thereof, in the past [3–6]; we, in particular, have shown that it may be obtained either from a model based on family lepton-number symmetries [7] or from a model based on the discrete eight-element group D_4 [8].

We show in this letter that the matrix (1.1) may also be obtained from a model based on the smaller discrete group S_3, a group which has a long tradition in model building [9]. The model presented here also suggests a generalization of the matrix (1.1), wherein

$$\mathcal{M}_\nu^{-1} = \begin{pmatrix} r & s & s \\ s & pe^{i\psi} & q \\ s & q & pe^{-i\psi} \end{pmatrix}. \tag{1.2}$$

This generalization, while leading to neither $U_{e3} = 0$ nor $\theta_{23} = \pi/4$, seems interesting in itself.

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We note that, in [4], the mass matrix (1.1) has been generalized in such a way that θ_{23} differs from $\pi/4$, while $U_{e3} = 0$ remains intact; on the other hand, with matrix (1.2) — as we shall see later — the deviation of U_{e3} from zero is correlated with the deviation of θ_{23} from $\pi/4$.

2. The model

We work in the context of a non-supersymmetric $\mathrm{SU}(2)_L \times \mathrm{U}(1)$ framework. The three left-handed lepton $\mathrm{SU}(2)_L$ doublets are denoted by $D_{e,\mu,\tau}$. The three right-handed charged-lepton $\mathrm{SU}(2)_L$ singlets are e_R, μ_R and τ_R. We further introduce three $\mathrm{SU}(2)_L$ singlet right-handed neutrinos ν_{eR}, $\nu_{\mu R}$ and $\nu_{\tau R}$, in order to enable the seesaw mechanism for suppressing the light-neutrino masses [10]. In our model there are three Higgs $\mathrm{SU}(2)_L$ doublets $\phi_{1,2,3}$. In exact analogy to the D_4 model [8], we introduce a symmetry

$$\mathbb{Z}_2^{(\mathrm{aux})}: \quad \nu_{eR},\ \nu_{\mu R},\ \nu_{\tau R},\ \phi_1,\ e_R \ \text{change sign}\,. \tag{2.1}$$

Instead of *two real* neutral scalar $\mathrm{SU}(2)_L$ singlets, as in [8], we use *one complex* neutral scalar $\mathrm{SU}(2)_L$ singlet, χ. Again in exact analogy to [8], we define a symmetry

$$\mathbb{Z}_2^{(\mathrm{tr})}: \quad D_\mu \leftrightarrow D_\tau, \qquad \mu_R \leftrightarrow \tau_R\,, \qquad \nu_{\mu R} \leftrightarrow \nu_{\tau R}\,, \qquad \chi \to \chi^*\,, \qquad \phi_3 \to -\phi_3\,. \tag{2.2}$$

This $\mathbb{Z}_2^{(\mathrm{tr})}$ is the μ-τ interchange symmetry. The crucial difference between the D_4 model [8] and the present S_3 one is that, while in the D_4 model there was a symmetry $\mathbb{Z}_2^{(\tau)}$ which, together with $\mathbb{Z}_2^{(\mathrm{tr})}$, generated a group D_4, in the S_3 model we introduce instead a symmetry $\mathbb{Z}_3$ which, together with $\mathbb{Z}_2^{(\mathrm{tr})}$, generates a group S_3 [11]. With $\omega \equiv \exp(2i\pi/3)$, we impose

$$\mathbb{Z}_3: \quad \begin{array}{ll} D_\mu \to \omega D_\mu\,, & D_\tau \to \omega^2 D_\tau\,, \\ \mu_R \to \omega \mu_R\,, & \tau_R \to \omega^2 \tau_R\,, \\ \nu_{\mu R} \to \omega \nu_{\mu R}, & \nu_{\tau R} \to \omega^2 \nu_{\tau R}\,, \\ \chi \to \omega \chi\,, & \chi^* \to \omega^2 \chi^*\,. \end{array} \tag{2.3}$$

Thus, (D_μ, D_τ), (μ_R, τ_R), $(\nu_{\mu R}, \nu_{\tau R})$ and (χ, χ^*) are doublets of S_3. The Higgs $\mathrm{SU}(2)_L$ doublet ϕ_3 changes sign under the odd permutations of S_3, but stays invariant under the cyclic permutations.

The Yukawa lagrangian symmetric under $S_3 \times \mathbb{Z}_2^{(\mathrm{aux})}$ is

$$\begin{aligned} \mathcal{L}_\mathrm{Y} = &- \left[y_1 \bar{D}_e \nu_{eR} + y_2 \left(\bar{D}_\mu \nu_{\mu R} + \bar{D}_\tau \nu_{\tau R} \right) \right] \tilde{\phi}_1 - \\ &- y_3 \bar{D}_e e_R \phi_1 - y_4 \left(\bar{D}_\mu \mu_R + \bar{D}_\tau \tau_R \right) \phi_2 - y_5 \left(\bar{D}_\mu \mu_R - \bar{D}_\tau \tau_R \right) \phi_3 + \\ &+ y_\chi^*\, \nu_{eR}^T C^{-1} \left(\nu_{\mu R} \chi^* + \nu_{\tau R} \chi \right) + \frac{z_\chi^*}{2} \left(\nu_{\mu R}^T C^{-1} \nu_{\mu R} \chi + \nu_{\tau R}^T C^{-1} \nu_{\tau R} \chi^* \right) + \text{H.c.}\,, \end{aligned} \tag{2.4}$$

where $\tilde{\phi}_1 \equiv i\tau_2 \phi_1^*$. There is also an $S_3 \times \mathbb{Z}_2^{(\mathrm{aux})}$-invariant Majorana mass term

$$\mathcal{L}_\mathrm{M} = \frac{m^*}{2}\, \nu_{eR}^T C^{-1} \nu_{eR} + m'^* \nu_{\mu R}^T C^{-1} \nu_{\tau R} + \text{H.c.}\,. \tag{2.5}$$

The second term in the right-hand side of (2.5) differs, in a crucial fashion, from the analogous term in the D_4 model — see equation (9) of [8].

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With vacuum expectation values (VEVs) $\left\langle 0 \left| \phi_j^0 \right| 0 \right\rangle = v_j$ for $j = 1, 2, 3$, one obtains

$$\begin{aligned} m_e &= |y_3 v_1| \,, \\ m_\mu &= |y_4 v_2 + y_5 v_3| \,, \\ m_\tau &= |y_4 v_2 - y_5 v_3| \,. \end{aligned} \tag{2.6}$$

The μ-τ interchange symmetry $\mathbb{Z}_2^{(\mathrm{tr})}$ is spontaneously broken by the VEV of ϕ_3^0, so that the μ and τ charged leptons acquire different masses. The smallness of m_μ relative to m_τ may be explained by requiring the model to be invariant under an additional, softly broken symmetry [12].

The neutrino Dirac mass matrix is

$$M_D = \mathrm{diag}\,(a, b, b) \,, \qquad \text{with } a = y_1^* v_1 \,,\ b = y_2^* v_1 \,. \tag{2.7}$$

When the singlet χ acquires a VEV $\langle 0 \,| \chi |\, 0 \rangle = W$, one obtains Majorana mass terms for the right-handed neutrinos:

$$\mathcal{L}_{M_R} = -\frac{1}{2} \left(\bar{\nu}_{eR},\ \bar{\nu}_{\mu R},\ \bar{\nu}_{\tau R} \right) M_R \, C \begin{pmatrix} \bar{\nu}_{eR}^T \\ \bar{\nu}_{\mu R}^T \\ \bar{\nu}_{\tau R}^T \end{pmatrix} + \mathrm{H.c.} \,, \tag{2.8}$$

with

$$M_R = \begin{pmatrix} m & y_\chi W & y_\chi W^* \\ y_\chi W & z_\chi W^* & m' \\ y_\chi W^* & m' & z_\chi W \end{pmatrix} . \tag{2.9}$$

We next perform a rephasing of the fields,

$$\begin{aligned} &\nu_{\mu R} \to e^{i\alpha} \nu_{\mu R} \,, \qquad D_\mu \to e^{i\alpha} D_\mu \,, \qquad \mu_R \to e^{i\alpha} \mu_R \,, \\ &\nu_{\tau R} \to e^{-i\alpha} \nu_{\tau R} \,, \qquad D_\tau \to e^{-i\alpha} D_\tau \,, \qquad \tau_R \to e^{-i\alpha} \tau_R \,, \qquad \text{with } \alpha \equiv \arg W \,, \end{aligned} \tag{2.10}$$

to obtain

$$M_R = \begin{pmatrix} m & y_\chi \,|W| & y_\chi \,|W| \\ y_\chi \,|W| & z_\chi \,|W| \, e^{-3i\alpha} & m' \\ y_\chi \,|W| & m' & z_\chi \,|W| \, e^{3i\alpha} \end{pmatrix} . \tag{2.11}$$

We see that the matrix M_R has become μ-τ symmetric after the rephasing, provided W^3 is real ($e^{3i\alpha} = \pm 1$). We shall see shortly that it is indeed possible to enforce this. If M_R is μ-τ symmetric, and since M_D also enjoys the μ-τ interchange symmetry, it follows, by applying the seesaw mechanism,[1] that

$$\mathcal{M}_\nu = -M_D^T M_R^{-1} M_D \tag{2.12}$$

is μ-τ symmetric, i.e. it is of the form (1.1).

We thus find that it is possible to produce a neutrino mass matrix of the form (1.1), which leads to $U_{e3} = 0$ and $\theta_{23} = \pi/4$, out of a model with symmetry $S_3 \times \mathbb{Z}_2^{(\mathrm{aux})}$ with

[1] We assume that m, m' and the VEV W are all of the same very large order of magnitude.

three Higgs $\mathrm{SU}(2)_L$ doublets — two of which are S_3-invariant while the third one changes sign under the odd permutations of S_3. The charged-lepton mass matrix is automatically diagonal, hence there are no flavour-changing neutral currents *at tree level* in the charged-lepton sector — such interactions appear, though, already at the one-loop level [13].

3. The scalar potential

Because of the symmetry $S_3 \times \mathbb{Z}_2^{(\mathrm{aux})}$, the scalar potential is

$$\begin{aligned} V =& \mu_\chi |\chi|^2 + \lambda |\chi|^4 + \sum_{j=1}^{3} \left(\phi_j^\dagger \phi_j\right) \left(\mu_j + a_j \phi_j^\dagger \phi_j + b_j |\chi|^2\right) + \\ &+ \sum_{j<k} \left[a_{jk} \left(\phi_j^\dagger \phi_j\right) \left(\phi_k^\dagger \phi_k\right) + b_{jk} \left(\phi_j^\dagger \phi_k\right) \left(\phi_k^\dagger \phi_j\right) + c_{jk} \left(\phi_j^\dagger \phi_k\right)^2 + c_{jk}^* \left(\phi_k^\dagger \phi_j\right)^2 \right] + \\ &+ m_\chi \left(\chi^3 + \chi^{*3}\right). \end{aligned} \tag{3.1}$$

Only the term in the last line of (3.1) feels the phase of χ. If its coefficient m_χ is negative, then the phase of the VEV W will adjust so that W^3 is real and positive, i.e. α will be either 0 or $\pm 2\pi/3$; if m_χ is positive, then α will be either π or $\pm\pi/3$, in order that W^3 is real and negative. In any case, W^3 is real. This is precisely what is needed in order to obtain a μ-τ-symmetric $\mathcal{M}_\nu$.

The situation is modified if we allow the symmetry $\mathbb{Z}_3$ of (2.3) to be softly broken by terms of dimension one, or one and two, while keeping both $\mathbb{Z}_2^{(\mathrm{aux})}$ and $\mathbb{Z}_2^{(\mathrm{tr})}$ unscathed.[2] There are only two such terms, namely

$$\mu'_\chi \left(\chi^2 + \chi^{*2}\right) + M \left(\chi + \chi^*\right), \tag{3.2}$$

with real constants μ'_χ, M. These terms get added to V in (3.1), which does not change otherwise. The phase of W becomes arbitrary. The matrix M_R in (2.11) does not respect μ-τ interchange symmetry any more, rather only a partial version thereof.

If one worries about cosmological domain walls, then one may want to eliminate from the lagrangian all exact discrete symmetries. This one may do by breaking $\mathbb{Z}_2^{(\mathrm{aux})}$ and $\mathbb{Z}_2^{(\mathrm{tr})}$, together with $\mathbb{Z}_3$, softly by terms of dimension two. This amounts to the addition, to the scalar potential (3.1), of all terms $\phi_j^\dagger \phi_k$ with $j \neq k$. (When $\mathbb{Z}_2^{(\mathrm{tr})}$ is broken softly, the terms of (3.2) also have to be generalized to $\mu'_\chi \chi^2 + M\chi + \mathrm{H.c.}$ with complex μ'_χ, M.) However, this soft breaking only affects the values of the v_j and has no influence on the lepton mass matrices. It is thus irrelevant for the following discussion.

4. Reproducing the D_4 model

In the D_4 model [8] there is an *accidental* symmetry

$$\mathbb{Z}_2^{(e)}: \quad D_e,\ e_R,\ \nu_{eR},\ \chi \text{ change sign}. \tag{4.1}$$

[2]If we also allow the soft breaking of (2.3) by terms of dimension three, then Majorana mass terms $\nu_{eR}^T C^{-1} \left(\nu_{\mu R} + \nu_{\tau R}\right)$ and $\nu_{\mu R}^T C^{-1} \nu_{\mu R} + \nu_{\tau R}^T C^{-1} \nu_{\tau R}$ are also present in the lagrangian and, after χ gets a VEV, the μ-τ interchange symmetry is destroyed altogether.

In the context of the present S_3 model, one may promote that symmetry to *fundamental* and impose it on the lagrangian from the start. It enforces $z_\chi = 0$, hence $(M_R)_{22} = (M_R)_{33} = 0$; since $M_R = -M_D \mathcal{M}_\nu^{-1} M_D^T$ and M_D is diagonal, this means the vanishing of the (μ, μ) and (τ, τ) matrix elements of $\mathcal{M}_\nu^{-1}$. The phase α of W is irrelevant when $(M_R)_{22} = (M_R)_{33} = 0$, since it may be rephased away as in (2.10), and the model is automatically μ-τ symmetric. Thus, in the S_3 model with the extra $\mathbb{Z}_2^{(e)}$ symmetry one has

$$\mathcal{M}_\nu^{-1} = \begin{pmatrix} r & s & s \\ s & 0 & q \\ s & q & 0 \end{pmatrix}, \tag{4.2}$$

i.e. $\left(\mathcal{M}_\nu^{-1}\right)_{\mu\mu} = \left(\mathcal{M}_\nu^{-1}\right)_{\tau\tau} = 0$.

On the other hand, in the D_4 model [8] one has $\left(\mathcal{M}_\nu^{-1}\right)_{\mu\tau} = 0$ but $\left(\mathcal{M}_\nu^{-1}\right)_{\mu\mu} = \left(\mathcal{M}_\nu^{-1}\right)_{\tau\tau} \neq 0$. We shall demonstrate now that the matrix (4.2) is equivalent to the mass matrix of the D_4 model.

In general, $\mathcal{M}_\nu$ is diagonalized as

$$U^T \mathcal{M}_\nu U = \mathrm{diag}\,(m_1, m_2, m_3)\,, \tag{4.3}$$

where U is the lepton mixing matrix and the m_j are the (real and non-negative) neutrino masses. Equivalently,

$$\mathcal{M}_\nu^{-1} = U\,\mathrm{diag}\left(m_1^{-1}, m_2^{-1}, m_3^{-1}\right) U^T\,. \tag{4.4}$$

The unitary U is parametrized as

$$\begin{aligned} U =\ & \mathrm{diag}\left(e^{i\vartheta_1}, e^{i\vartheta_2}, e^{i\vartheta_3}\right) \times \begin{pmatrix} c_{13}c_{12} & c_{13}s_{12} & s_{13}e^{-i\delta} \\ -c_{23}s_{12} - s_{23}s_{13}c_{12}e^{i\delta} & c_{23}c_{12} - s_{23}s_{13}s_{12}e^{i\delta} & s_{23}c_{13} \\ -s_{23}s_{12} + c_{23}s_{13}c_{12}e^{i\delta} & s_{23}c_{12} + c_{23}s_{13}s_{12}e^{i\delta} & -c_{23}c_{13} \end{pmatrix} \times \\ & \times \mathrm{diag}\left(e^{i\Delta/2}, 1, e^{i\Omega/2}\right), \end{aligned} \tag{4.5}$$

where $c_{ij} \equiv \cos\theta_{ij}$, $s_{ij} \equiv \sin\theta_{ij}$ and θ_{12} is the solar mixing angle. The phases ϑ_i are unphysical; physical are only the Dirac phase δ and the Majorana phases Δ and Ω. In the case of μ-τ symmetric $\mathcal{M}_\nu^{-1}$, one has — see for instance [14] — $\vartheta_2 = \vartheta_3$, $\theta_{13} = 0$ and $\theta_{23} = \pi/4$; the vanishing of θ_{13} allows one to write the non-diagonal matrix on the right-hand side of (4.5) as a product $U_{23}U_{12}$, where U_{23} and U_{12} are responsible for the mixing in the atmospheric and solar neutrino sector, respectively.

Now we turn to the matrix (4.2), which we can transform according to

$$T\mathcal{M}_\nu^{-1}T = \begin{pmatrix} r & s & s \\ s & q & 0 \\ s & 0 & q \end{pmatrix}, \qquad \text{with} \qquad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & u & u^* \\ 0 & u^* & u \end{pmatrix} \qquad \text{and} \qquad u = \frac{e^{i\pi/4}}{\sqrt{2}}\,. \tag{4.6}$$

Thus, $T\mathcal{M}_\nu^{-1}T$ has precisely the form of the mass matrix of the D_4 model, and from (4.4) it is clear that it can be diagonalized by a U appropriate for a μ-τ symmetric matrix. Since

$\vartheta_2 = \vartheta_3$, in the diagonalization of $T\mathcal{M}_\nu^{-1}T$ the product TU_{23} occurs. Now,

$$TU_{23} = U_{23}\,\mathrm{diag}\,(1,1,i)\,, \qquad \text{where} \qquad U_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho & \rho \\ 0 & \rho & -\rho \end{pmatrix} \qquad \text{and} \qquad \rho = \frac{1}{\sqrt{2}}\,. \tag{4.7}$$

The matrix U_{12} commutes with $\mathrm{diag}\,(1,1,i)$. Therefore, the difference between the D_4 model and the $S_3 \times \mathbb{Z}_2^{(\mathrm{aux})} \times \mathbb{Z}_2^{(\mathrm{e})}$ model amounts to the modification $\Omega \to \Omega + \pi$. Since Ω is a free parameter, the two models are in practice equivalent.

Just as in the D_4 model [8], the zero in the matrix (4.2), or — equivalently — the zero in $T\mathcal{M}_\nu^{-1}T$ of (4.6), leads to

$$\frac{s_{12}^2\,e^{i\Delta}}{m_1} + \frac{c_{12}^2}{m_2} + \frac{e^{i\Omega}}{m_3} = 0\,. \tag{4.8}$$

As we have shown in [8], the constraint (4.8) implies, given the known experimental values, a normal mass spectrum $m_1 < m_2 < m_3$, with m_1 either in the range 3 to $9{\times}10^{-3}\,$eV, or larger than $14{\times}10^{-3}\,$eV; these numbers hold for the best-fit values of the mass-squared differences as given in [1]. A further prediction is $|\langle m\rangle| = m_1 m_2/m_3$, where $|\langle m\rangle|$ is the effective mass relevant for neutrinoless $\beta\beta$ decay.

5. Generalization of the μ-τ interchange symmetry

We now abandon the symmetry $\mathbb{Z}_2^{(\mathrm{e})}$ and return to the general M_R of the S_3 model, given in (2.11). Since $\mathcal{M}_\nu^{-1} = -M_D^{-1} M_R M_D^{T-1}$, and since $M_D = \mathrm{diag}\,(a,b,b)$ is diagonal, one obtains $\mathcal{M}_\nu^{-1}$ of the form (1.2).

In general, the symmetric matrix $\mathcal{M}_\nu^{-1}$ contains nine parameters: the six moduli of its matrix elements and three rephasing-invariant phases, because one may independently rephase the three left-handed neutrinos, thereby eliminating three phases in $\mathcal{M}_\nu$, or equivalently in $\mathcal{M}_\nu^{-1}$. To those nine parameters correspond nine observables: the three neutrino masses $m_{1,2,3}$, the three mixing angles $\theta_{12,13,23}$, the Dirac phase δ and the Majorana phases Δ and Ω.

In the case of (full) μ-τ interchange symmetry, i.e. when $e^{i\psi} = \pm 1$ in (1.2), three observables are predicted: $\theta_{23} = \pi/4$, $\theta_{13} = 0$ and the Dirac phase is meaningless because $\theta_{13} = 0$. To those three predicted observables correspond three rephasing-invariant relations among the parameters of $\mathcal{M}_\nu$, or of $\mathcal{M}_\nu^{-1}$:

$$\left| \left(\mathcal{M}_\nu^{-1}\right)_{e\mu} \right| = \left| \left(\mathcal{M}_\nu^{-1}\right)_{e\tau} \right|, \tag{5.1}$$

$$\left| \left(\mathcal{M}_\nu^{-1}\right)_{\mu\mu} \right| = \left| \left(\mathcal{M}_\nu^{-1}\right)_{\tau\tau} \right|, \tag{5.2}$$

$$\arg\left\{ \left[\left(\mathcal{M}_\nu^{-1}\right)_{e\mu} \right]^2 \left(\mathcal{M}_\nu^{-1}\right)_{\tau\tau} \right\} = \arg\left\{ \left[\left(\mathcal{M}_\nu^{-1}\right)_{e\tau} \right]^2 \left(\mathcal{M}_\nu^{-1}\right)_{\mu\mu} \right\}. \tag{5.3}$$

In the case of the matrix (1.2), the condition (5.3) does not apply. One has an *incomplete μ-τ interchange symmetry*, wherein conditions (5.1) and (5.2) apply, but not condition (5.3). The matrix (1.2) has seven real physical parameters. As we will see, $\psi \neq 0$

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leads both $\cos 2\theta_{23}$ and s_{13} to be non-zero, and in general there will also be a non-zero Dirac phase δ. Since the matrix (1.2) has only one parameter more than matrix (1.1), it must predict two relations: two of the observables $\cos 2\theta_{23}$, s_{13} and δ must be functions of the third one and of the remaining observables, which are $m_{1,2,3}$, θ_{12}, Δ and Ω. Since the Majorana phases are hardly accessible by experiment, our aim is to derive observable consequences of incomplete μ-τ interchange symmetry which do not involve those phases.

It is convenient to use the facts that, from experiment, it is known that the atmospheric mixing angle θ_{23} is close to $\pi/4$ and that θ_{13} is small. Thus we define the parameters

$$\nu \equiv \cos 2\theta_{23}\,, \tag{5.4}$$

$$\epsilon \equiv s_{13} e^{i\delta}\,, \tag{5.5}$$

the latter of which is complex. Experimentally [1], $|\nu| < 0.28$ at 90% confidence level and $|\epsilon| < 0.22$ at 3σ level. In the case of full μ-τ interchange symmetry, $\nu = \epsilon = 0$ and there are no restrictions on all other observables, i.e. on $m_{1,2,3}$, θ_{12}, Δ and Ω. In the case of incomplete μ-τ interchange symmetry, ν and ϵ in general do not vanish and, when they are non-zero, some restrictions may apply on the other observables.

Adding (5.1) and (5.2) and using (4.4), one finds that

$$\begin{aligned}
0 &= \left|\left(\mathcal{M}_\nu^{-1}\right)_{e\mu}\right|^2 + \left|\left(\mathcal{M}_\nu^{-1}\right)_{\mu\mu}\right|^2 - \left|\left(\mathcal{M}_\nu^{-1}\right)_{e\tau}\right|^2 - \left|\left(\mathcal{M}_\nu^{-1}\right)_{\tau\tau}\right|^2 && (5.6)\\
&= \left(\mathcal{M}_\nu^{-1}\mathcal{M}_\nu^{-1*}\right)_{\mu\mu} - \left(\mathcal{M}_\nu^{-1}\mathcal{M}_\nu^{-1*}\right)_{\tau\tau} && (5.7)\\
&= \sum_{j=1}^{3} \frac{|U_{\mu j}|^2 - |U_{\tau j}|^2}{m_j^2} && (5.8)\\
&= \left(c_{23}^2 - s_{23}^2\right)\left(\frac{s_{12}^2 - s_{13}^2 c_{12}^2}{m_1^2} + \frac{c_{12}^2 - s_{13}^2 s_{12}^2}{m_2^2} - \frac{c_{13}^2}{m_3^2}\right) + \\
&\quad + 4\left(\frac{1}{m_1^2} - \frac{1}{m_2^2}\right) c_{23} s_{23} s_{13} c_{12} s_{12} \cos\delta\,. && (5.9)
\end{aligned}$$

This condition is particularly useful since it does not involve the Majorana phases. It translates into

$$\left(\frac{s_{12}^2 - |\epsilon|^2 c_{12}^2}{m_1^2} + \frac{c_{12}^2 - |\epsilon|^2 s_{12}^2}{m_2^2} + \frac{|\epsilon|^2 - 1}{m_3^2}\right)\nu + 2\left(\frac{1}{m_1^2} - \frac{1}{m_2^2}\right) c_{12} s_{12} \sqrt{1-\nu^2}\,\mathrm{Re}\,\epsilon = 0\,. \tag{5.10}$$

Numerically, we shall use (5.10) to determine ν as a function of ϵ, for various values of the neutrino masses and of the mixing angle θ_{12}. Since $|\epsilon|^2$ and ν^2 are in any case rather small, (5.10) is an almost linear relationship between ν and $\mathrm{Re}\,\epsilon$.

We next consider the constraint (5.1) by itself alone. It is equivalent to the existence of a phase φ such that

$$\begin{aligned}
0 &= \left(\mathcal{M}_\nu^{-1}\right)_{e\mu} - e^{i\varphi}\left(\mathcal{M}_\nu^{-1}\right)_{e\tau} && (5.11)\\
&= \sum_{j=1}^{3} \frac{U_{ej}\left(U_{\mu j} - e^{i\varphi} U_{\tau j}\right)}{m_j}\,. && (5.12)
\end{aligned}$$

This is the equation of a triangle in the complex plane — it states that the sum of three complex numbers vanishes, i.e. that those three numbers form a triangle in the complex plane. The triangle (5.12) involves the Majorana phases. It is convenient to remove those phases, since they are in practice very difficult to observe experimentally. One does this by considering the inequality, which follows from (5.12),[3]

$$\sum_{j=1}^{3} \frac{\left|U_{ej}\left(U_{\mu j}-e^{i\varphi}U_{\tau j}\right)\right|^4}{m_j^4} - 2\sum_{j<k} \frac{\left|U_{ej}\left(U_{\mu j}-e^{i\varphi}U_{\tau j}\right)\right|^2 \left|U_{ek}\left(U_{\mu k}-e^{i\varphi}U_{\tau k}\right)\right|^2}{m_j^2 m_k^2} \leq 0\,. \quad (5.13)$$

Notice that, using (4.5), one has

$$\begin{aligned} \frac{\left|U_{e1}\left(U_{\mu 1}-e^{i\varphi}U_{\tau 1}\right)\right|^2}{c_{13}^2} &= c_{12}^2 s_{12}^2 \left(1-\sqrt{1-\nu^2}\cos\phi\right) + c_{12}^4 |\epsilon|^2 \left(1+\sqrt{1-\nu^2}\cos\phi\right) + \\ &\quad + 2c_{12}^3 s_{12} \left(\nu \operatorname{Re}\epsilon\cos\phi - \operatorname{Im}\epsilon\sin\phi\right), \quad (5.14) \end{aligned}$$

$$\begin{aligned} \frac{\left|U_{e2}\left(U_{\mu 2}-e^{i\varphi}U_{\tau 2}\right)\right|^2}{c_{13}^2} &= c_{12}^2 s_{12}^2 \left(1-\sqrt{1-\nu^2}\cos\phi\right) + s_{12}^4 |\epsilon|^2 \left(1+\sqrt{1-\nu^2}\cos\phi\right) - \\ &\quad - 2c_{12} s_{12}^3 \left(\nu \operatorname{Re}\epsilon\cos\phi - \operatorname{Im}\epsilon\sin\phi\right), \quad (5.15) \end{aligned}$$

$$\frac{\left|U_{e3}\left(U_{\mu 3}-e^{i\varphi}U_{\tau 3}\right)\right|^2}{c_{13}^2} = |\epsilon|^2 \left(1+\sqrt{1-\nu^2}\cos\phi\right), \quad (5.16)$$

where $\phi \equiv \varphi + \vartheta_3 - \vartheta_2$.

Numerically, we use (5.10) to determine ν as a function of ϵ, for various values of the neutrino masses and of the mixing angle θ_{12}. Afterwards, we check whether there is any phase φ for which the inequality (5.13) is satisfied. If there is, then those values of the neutrino masses, mixing angles and Dirac phase are compatible with incomplete μ-τ interchange symmetry; otherwise they are not. For simplicity we keep $\theta_{12} = 33°$, $m_2^2 - m_1^2 = 8.1 \times 10^{-5}\,\mathrm{eV}^2$ and $\left|m_3^2 - m_1^2\right| = 2.2 \times 10^{-3}\,\mathrm{eV}^2$ fixed at their best-fit values [1]. It is important to remark that (5.14)–(5.16), just as (5.10), are symmetric under $\nu \to -\nu, \operatorname{Re}\epsilon \to -\operatorname{Re}\epsilon$. This means that one only has to study the region of positive $\operatorname{Re}\epsilon$. Also, (5.14)–(5.16) are invariant under $\operatorname{Im}\epsilon \to -\operatorname{Im}\epsilon$, $\sin\phi \to -\sin\phi$. This means that we only have to consider positive values of $\operatorname{Im}\epsilon$, provided we test all possible values of ϕ.

In the case where $m_3^2 - m_1^2 < 0$, the situation is rather simple and it is aptly described by figure 1. The parameter ν has the same sign as $\operatorname{Re}\epsilon$ but it is much smaller in absolute value; the atmospheric mixing angle is, for all practical purposes, maximal; in the limit of very small m_3 the relation $\theta_{23} = 45°$ becomes exact — see (5.10) and figure 1. The exact value of $\operatorname{Im}\epsilon$ is practically immaterial in the determination of ν as a function of $\operatorname{Re}\epsilon$. The inequality (5.13) is always satisfied, hence it has no bearing on the overall picture.

[3] It is possible to construct a triangle with sides of (real, non-negative) lengths a, b and c if and only if

$$a \leq b+c, \qquad b \leq a+c \qquad \text{and} \qquad c \leq a+b.$$

It is easily shown that this set of three inequalities is equivalent to the sole inequality [15]

$$a^4 + b^4 + c^4 - 2\left(a^2 b^2 + a^2 c^2 + b^2 c^2\right) \leq 0\,.$$

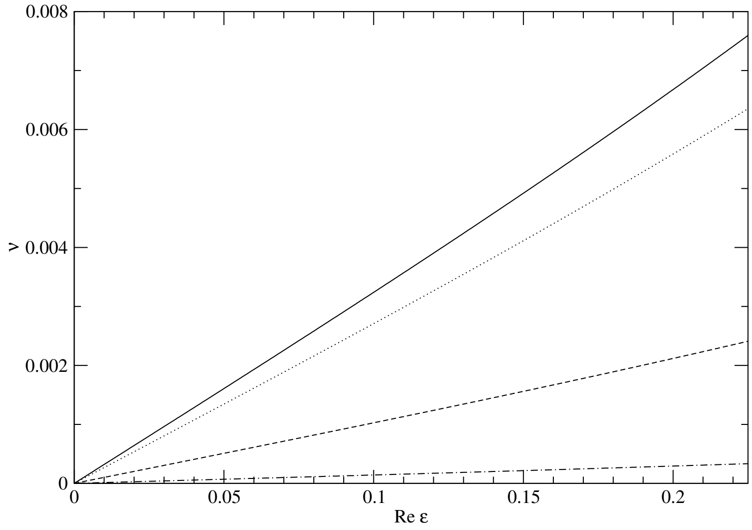

Figure 1: ν as a function of $\mathrm{Re}\,\epsilon$ in the case $m_3 < m_{1,2}$. The figure corresponds to a vanishing Im ϵ, but it would be practically identical for any Im ϵ of order 0.1 or smaller. The full line is for $m_3^2 = 10^{-1}\,\mathrm{eV}^2$, the dotted line for $m_3^2 = 10^{-2}\,\mathrm{eV}^2$, the dashed line for $m_3^2 = 10^{-3}\,\mathrm{eV}^2$ and the dashed-dotted line for $m_3^2 = 10^{-4}\,\mathrm{eV}^2$.

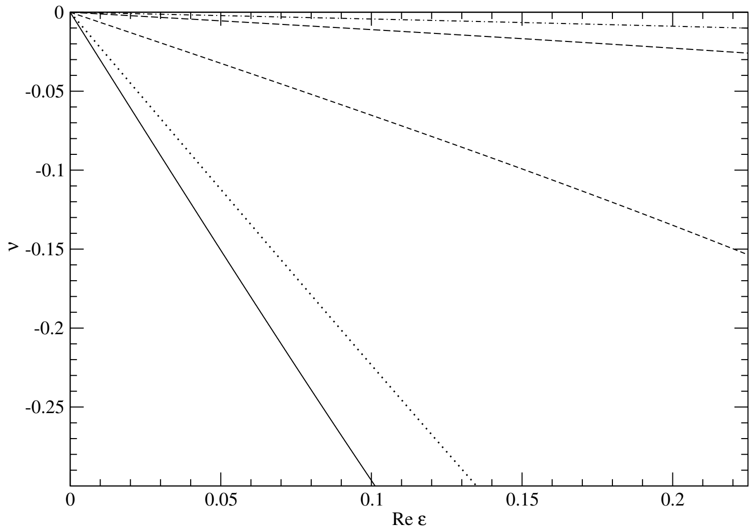

Figure 2: ν as a function of $\mathrm{Re}\,\epsilon$ in the case $m_3 > m_{1,2}$ and with Im $\epsilon = 0.1$ (a smaller Im ϵ yields practically the same curves, except for the exclusion zones depicted in figure 3). The full line is for $m_1^2 = 10^{-6}\,\mathrm{eV}^2$, the dotted line for $m_1^2 = 10^{-5}\,\mathrm{eV}^2$, the small-dashed line for $m_1^2 = 10^{-4}\,\mathrm{eV}^2$, the large-dashed line for $m_1^2 = 10^{-3}\,\mathrm{eV}^2$ and the dashed-dotted line for $m_1^2 = 10^{-2}\,\mathrm{eV}^2$.

In the case where $m_3^2 - m_1^2 > 0$ the situation is different. The parameters ν and $\mathrm{Re}\,\epsilon$ have opposite signs and ν is not necessarily small. On the other hand, the determination of ν as a function of $\mathrm{Re}\,\epsilon$ is, once again, largely insensitive to the exact value of Im ϵ. Typical values are displayed in figure 2.

When $m_3^2 - m_1^2 > 0$, inequality (5.13) introduces a complication because in this case there are values of the pair (ν, ϵ) for which that inequality is satisfied by no phase φ at

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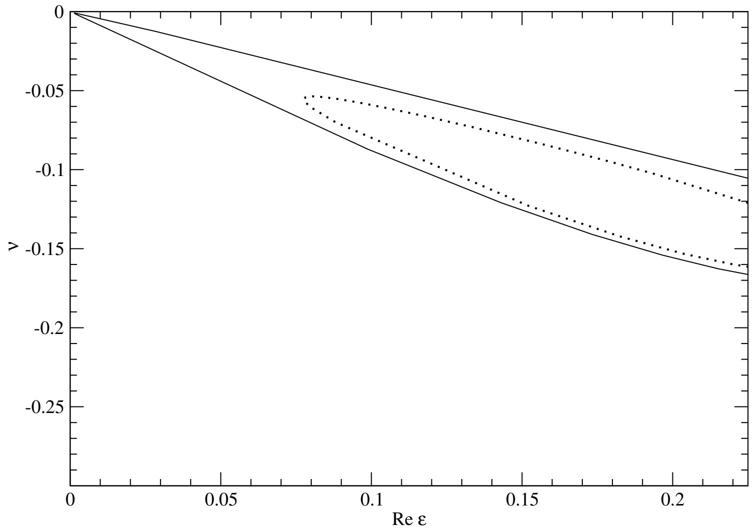

Figure 3: In the case $m_3 > m_{1,2}$, the region in the ν-$\mathrm{Re}\,\epsilon$ plane inside the full line is excluded by (5.13) when $\mathrm{Im}\ \epsilon = 10^{-4}$. The region inside the dotted line is excluded when $\mathrm{Im}\ \epsilon = 5 \times 10^{-3}$. For $\mathrm{Im}\ \epsilon$ larger than 10^{-2} there is no excluded region any more.

all. With ν and $\mathrm{Re}\,\epsilon$ obeying the relation (5.10), this happens when $\mathrm{Im}\ \epsilon \lesssim 0.01$ and $m_1 \sim 10^{-2}\,\mathrm{eV}$. For $\mathrm{Im}\ \epsilon = 10^{-4}$ and 5×10^{-3}, the corresponding excluded regions in the $(\nu, \mathrm{Re}\,\epsilon)$ plane are depicted in figure 3. That figure should be superimposed on figure 2 in order to see which curves or which parts of the curves in that figure are excluded and to find out the range of values of m_1^2 for which an excluded region arises.[4] Of the curves depicted in figure 2, only the small-dashed line, referring to $m_1 = 10^{-2}\,\mathrm{eV}$, is affected: for $\mathrm{Im}\ \epsilon = 10^{-4}$ that line is almost completely excluded, whereas for $\mathrm{Im}\ \epsilon = 5 \times 10^{-3}$ it is excluded if $\mathrm{Re}\,\epsilon \gtrsim 0.08$. Explicitly, we have found that, when $\mathrm{Im}\ \epsilon$ vanishes, excluded values of ν and $\mathrm{Re}\,\epsilon$ arise for $7.80 \times 10^{-3}\,\mathrm{eV} < m_1 < 1.28 \times 10^{-2}\,\mathrm{eV}$; when $\mathrm{Im}\ \epsilon = 5 \times 10^{-3}$, excluded values of $\mathrm{Re}\,\epsilon$ arise only for m_1 in between $8.49 \times 10^{-3}\,\mathrm{eV}$ and $1.16 \times 10^{-2}\,\mathrm{eV}$.

The excluded regions in the $(\nu, \mathrm{Re}\,\epsilon)$ plane can be translated into lower bounds for the CP-violating phase δ. These lower bounds are functions of m_1 and of s_{13}. Taking $m_1 = 0.01\,\mathrm{eV}$, i.e. m_1 in the center of the range where excluded regions occur, we numerically find $\delta \gtrsim 2.44°$ for $s_{13} = 0.2$, $\delta \gtrsim 3.34°$ for $s_{13} = 0.1$, $\delta \gtrsim 3.52°$ for $s_{13} = 0.01$, and $\delta \gtrsim 3.83°$ for $s_{13} = 0.001$. (We have confined ourselves to δ in the first quadrant, i.e. to the real and imaginary part of ϵ being both positive. The bounds on δ in the first quadrant get transferred into the other quadrants by using the symmetries $\mathrm{Re}\,\epsilon \to -\mathrm{Re}\,\epsilon$, $\nu \to -\nu$ and $\mathrm{Im}\ \epsilon \to -\mathrm{Im}\ \epsilon$, $\sin\phi \to -\sin\phi$ referred to earlier.) One sees that the excluded domain is hardly significant in terms of δ. For $s_{13} \gtrsim 0.1$, this is qualitatively understandable from the fact that for $\mathrm{Im}\ \epsilon = s_{13} \sin\delta \gtrsim 0.01$ there is no exclusion region anymore.

[4]In this comparison we use the fact that the curves in figure 2 are practically independent of the value of $\mathrm{Im}\ \epsilon$, provided $\mathrm{Im}\ \epsilon \lesssim 0.1$.

6. Conclusions

In this paper we have considered an extension of the Standard Model based on the horizontal symmetry group $S_3 \times \mathbb{Z}_2$, the seesaw mechanism and a complex scalar gauge singlet χ. Though S_3 is a time-honoured symmetry, the new feature here is the use of the complex scalar gauge singlet, with (χ, χ^*) transforming as a two-dimensional irreducible representation of S_3 — see (2.2) and (2.3). The gauge multiplets of our extension are those of the Standard Model, supplemented by two additional Higgs doublets, the scalar singlet and three right-handed neutrino singlets for the seesaw mechanism. The horizontal symmetry enforces diagonal charged-lepton and neutrino Dirac mass matrices. Our model has some freedom with regard to the realization of the symmetry $S_3 \times \mathbb{Z}_2$, and this freedom affects the Majorana mass matrix of the right-handed neutrinos. In this way, we are able to recover two mass matrices already found in the literature, derived from different horizontal symmetries, and also the mass matrix (1.2), a generalization thereof; we regard this flexibility as the distinguishing feature of the $S_3 \times \mathbb{Z}_2$ model. In terms of the inverted light-neutrino mass matrix (1.2), our results can be described in the following way:

1. Imposing the additional discrete electron number $\mathbb{Z}_2^{(e)}$ of (4.1), one obtains $\psi = 0$ and $q = 0$; one thus recovers a mass matrix originally derived in [8] from a horizontal symmetry group D_4.

2. Without $\mathbb{Z}_2^{(e)}$ and with exact $S_3 \times \mathbb{Z}_2$ symmetry of the lagrangian, one gets $\psi = 0$, i.e. the μ-τ symmetric mass matrix originally obtained in [7] in a framework of softly broken lepton numbers.

3. Breaking $S_3 \times \mathbb{Z}_2$ softly in the scalar potential, one obtains the matrix (1.2) without further restrictions.

The first and second realizations have a μ-τ symmetric mass matrix, with the well-known predictions of maximal atmospheric neutrino mixing and vanishing mixing-matrix element U_{e3}. If, in addition, $q = 0$ holds, then the model becomes much more predictive: it requires a normal neutrino mass ordering $m_1 < m_2 < m_3$, and the effective mass in neutrinoless $\beta\beta$ decay is a simple function of the neutrino masses alone [8]. Below the seesaw scale, the first and second realizations cannot be distinguished from the models in [8] and [7], respectively.

With $\psi \neq 0$, the mass matrix (1.2) has seven physical parameters and partially breaks the μ-τ interchange symmetry; the matrix of the absolute values of the elements of the *inverted* neutrino mass matrix (1.2) is still μ-τ symmetric.[5] In contrast to full μ-τ interchange symmetry, this partial symmetry induces non-zero $\cos 2\theta_{23}$ and s_{13}, and CP violation in neutrino mixing via the Dirac phase δ. These three quantities are functions of ψ. Fixing the neutrino masses and the solar mixing angle, there is an almost linear relation (5.10) between $\cos 2\theta_{23}$ and $s_{13} \cos \delta$, which is not obfuscated by the Majorana phases. This relation differs substantially depending on the type of neutrino mass spectrum: in

[5] Any mass matrix with that property can be transformed into (1.2) by a phase transformation.

the inverted case, atmospheric mixing is always maximal for all practical purposes, even when s_{13} is close to its experimental upper bound; in the normal case, a large $s_{13} \cos \delta$ is correlated with a large $\cos 2\theta_{23}$ with an opposite sign.

Finally, as an additional virtue, we mention that, for a normal neutrino mass spectrum, leptogenesis can naturally be accommodated in the present model with the μ-τ symmetric mass matrices [14]; at least for small s_{13}, the same must hold with partial μ-τ interchange symmetry.

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References

[1] For a review, see for instance M. Maltoni, T. Schwetz, M.A. Tortola and J.W.F. Valle, *Status of global fits to neutrino oscillations*, *New J. Phys.* **6** (2004) 122 [hep-ph/0405172].

[2] T. Fukuyama and H. Nishiura, *Mass matrix of Majorana neutrinos*, hep-ph/9702253;
E. Ma and M. Raidal, *Neutrino mass, muon anomalous magnetic moment and lepton flavor nonconservation*, *Phys. Rev. Lett.* **87** (2001) 011802 [hep-ph/0102255];
C.S. Lam, *A 2-3 symmetry in neutrino oscillations*, *Phys. Lett.* **B 507** (2001) 214 [hep-ph/0104116];
K.R.S. Balaji, W. Grimus and T. Schwetz, *The solar LMA neutrino oscillation solution in the Zee model*, *Phys. Lett.* **B 508** (2001) 301 [hep-ph/0104035];
E. Ma, *The all-purpose neutrino mass matrix*, *Phys. Rev.* **D 66** (2002) 117301 [hep-ph/0207352];
P.F. Harrison and W.G. Scott, *$\mu - \tau$ reflection symmetry in lepton mixing and neutrino oscillations*, *Phys. Lett.* **B 547** (2002) 219 [hep-ph/0210197].

[3] S.K. Kang and C.S. Kim, *Majorana neutrino masses and neutrino oscillations*, *Phys. Rev.* **D 63** (2001) 113010 [hep-ph/0012046];
T. Kitabayashi and M. Yasue, *Large solar neutrino mixing and radiative neutrino mechanism*, *Phys. Lett.* **B 524** (2002) 308 [hep-ph/0110303]; *S_{2L} permutation symmetry for left-handed μ and τ families and neutrino oscillations in an $\mathrm{SU}(3)_L \times \mathrm{U}(1)_N$ gauge model*, *Phys. Rev.* **D 67** (2003) 015006 [hep-ph/0209294];
I. Aizawa, M. Ishiguro, T. Kitabayashi and M. Yasue, *Bilarge neutrino mixing and $\mu - \tau$ permutation symmetry for two-loop radiative mechanism*, *Phys. Rev.* **D 70** (2004) 015011 [hep-ph/0405201];
R.N. Mohapatra and S. Nasri, *Leptogenesis and $\mu - \tau$ symmetry*, *Phys. Rev.* **D 71** (2005) 033001 [hep-ph/0410369];
R.N. Mohapatra, S. Nasri and H.-B. Yu, *Leptogenesis, $\mu - \tau$ symmetry and θ_{13}*, *Phys. Lett.* **B 615** (2005) 231 [hep-ph/0502026];
C.S. Lam, *Neutrino $2-3$ symmetry and inverted hierarchy*, *Phys. Rev.* **D 71** (2005) 093001 [hep-ph/0503159].

[4] W. Grimus, A.S. Joshipura, S. Kaneko, L. Lavoura and M. Tanimoto, *Lepton mixing angle $\theta_{13} = 0$ with a horizontal symmetry D_4*, *JHEP* **07** (2004) 078 [hep-ph/0407112].

JHEP08(2005)013

[5] Y. Koide, H. Nishiura, K. Matsuda, T. Kikuchi and T. Fukuyama, *Universal texture of quark and lepton mass matrices and a discrete symmetry Z_3*, *Phys. Rev.* **D 66** (2002) 093006 [hep-ph/0209333];
Y. Koide, *Universal texture of quark and lepton mass matrices with an extended flavor $2 \leftrightarrow 3$ symmetry*, *Phys. Rev.* **D 69** (2004) 093001 [hep-ph/0312207];
K. Matsuda and H. Nishiura, *Prediction for quark mixing from universal quark and lepton mass matrices with flavor $2 \leftrightarrow 3$ symmetry*, *Phys. Rev.* **D 71** (2005) 073001 [hep-ph/0501201].

[6] I. Aizawa and M. Yasue, *General property of neutrino mass matrix and CP-violation*, *Phys. Lett.* **B 607** (2005) 267 [hep-ph/0409331].

[7] W. Grimus and L. Lavoura, *Softly broken lepton numbers and maximal neutrino mixing*, *JHEP* **07** (2001) 045 [hep-ph/0105212];
W. Grimus and L. Lavoura, *Softly broken lepton numbers: an approach to maximal neutrino mixing*, *Acta Phys. Polon.* **B32** (2001) 3719 [hep-ph/0110041].

[8] W. Grimus and L. Lavoura, *A discrete symmetry group for maximal atmospheric neutrino mixing*, *Phys. Lett.* **B 572** (2003) 189 [hep-ph/0305046].

[9] S. Pakvasa and H. Sugawara, *Discrete symmetry and Cabibbo angle*, *Phys. Lett.* **B 73** (1978) 61.

[10] P. Minkowski, *$\mu \to e\gamma$ at a rate of one out of 10^9 muon decays?*, *Phys. Lett.* **B 67** (1977) 421;
T. Yanagida, *Horizontal gauge symmetry and masses of neutrinos*, in *Proceedings of the workshop on unified theory and Baryon number in the universe*, O. Sawata and A. Sugamoto eds., KEK report **79-18**, Tsukuba, Japan 1979;
S.L. Glashow, in *Quarks and leptons, proceedings of the advanced study institute (Cargèse, Corsica, 1979)*, J.-L. Basdevant et al. eds., Plenum, New York, 1981;
M. Gell-Mann, P. Ramond and R. Slansky, *Complex spinors and unified theories*, in *Supergravity*, D.Z. Freedman and F. van Nieuwenhuizen eds., North Holland, Amsterdam 1979;
R.N. Mohapatra and G. Senjanovic, *Neutrino mass and spontaneous parity nonconservation*, *Phys. Rev. Lett.* **44** (1980) 912.

[11] For previous uses of the group S_3 in the context of lepton mixing, see E. Ma, *$S_3 \times Z_3$ model of lepton mass matrices*, *Phys. Rev.* **D 44** (1991) 587;
J. Kubo, A. Mondragon, M. Mondragon and E. Rodriguez-Jauregui, *The flavor symmetry*, *Prog. Theor. Phys.* **109** (2003) 795 [hep-ph/0302196];
J. Kubo, *Majorana phase in minimal S_3 invariant extension of the standard model*, *Phys. Lett.* **B 578** (2004) 156 [hep-ph/0309167];
S.-L. Chen, M. Frigerio and E. Ma, *Large neutrino mixing and normal mass hierarchy: a discrete understanding*, *Phys. Rev.* **D 70** (2004) 073008 [hep-ph/0404084], erratum *ibid.* **70** (2004) 079905.

[12] W. Grimus and L. Lavoura, *Maximal atmospheric neutrino mixing and the small ratio of muon to τ mass*, *J. Phys.* **G 30** (2004) 73 [hep-ph/0309050].

[13] W. Grimus and L. Lavoura, *Soft lepton-flavor violation in a multi-Higgs-doublet seesaw model*, *Phys. Rev.* **D 66** (2002) 014016 [hep-ph/0204070].

[14] W. Grimus and L. Lavoura, *Leptogenesis in seesaw models with a twofold-degenerate neutrino dirac mass matrix*, *J. Phys.* **G 30** (2004) 1073 [hep-ph/0311362];

W. Grimus and L. Lavoura, *Models of maximal atmospheric neutrino mixing and leptogenesis*, hep-ph/0405261.

[15] G.C. Branco and L. Lavoura, *Rephasing invariant parametrization of the quark mixing matrix*, *Phys. Lett.* **B 208** (1988) 123;
see also G.C. Branco, L. Lavoura and J.P. Silva, *CP violation*, Oxford University Press, Oxford 1999.